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## INTEGRATION OF RICCATI'S EQUATION.

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For illustrating important points of analysis, this Equation is adapted to a place in the Calculus, similar to that of the "problem of the lights" in Algebra. Its "integrable cases" were first pointed out by Count Riccati in the Leipsic Acts of 1732.

RICCATI'S EQUATION, $$\frac{du}{dx} + bu^2 = cx^m. \qquad (1)$$

### FIRST INTEGRABLE CASE.

When $m = 0$, the last term becomes $c$; and $\frac{du}{c - bu^2} = dx$. Integrating this, and then reducing, with $C$ as the arbitrary constant, and $\varepsilon$ as the base of hyperbolic logarithms, we obtain the integral (2) when $b$ and $c$ have like signs, and the integral (3) when their signs are unlike.

$$u = \left(\frac{c}{b}\right)^{\frac{1}{2}} \frac{C\varepsilon^{2x(bc)^{\frac{1}{2}}} - 1}{C\varepsilon^{2x(bc)^{\frac{1}{2}}} + 1}. \qquad (2)$$

$$u = \left(-\frac{c}{b}\right)^{\frac{1}{2}} \tan\left(C + x(-bc)^{\frac{1}{2}}\right). \qquad (3)$$

### SECOND INTEGRABLE CASE.

When $m = -2$, let the value $u = \frac{y}{x} + \frac{1}{2bx}$ be substituted in (1), which leads to the result (4), of which the integral $y$ is evidently found from (2) or (3) by changing $x$ into $\log x$, and $c$ into $c + \frac{1}{4b}$.

$$\frac{x\,dy}{dx} + by^2 = \left(c + \frac{1}{4b}\right), \quad u = \frac{y}{x} + \frac{1}{2bx}. \qquad (4)$$

### OTHER INTEGRABLE CASES.

By the reciprocal and direct transformations below, it was long since proved that Riccati's Equation is always integrable in a *continued fraction*, or in finite terms, when its exponent $m$ or $m'$ is a term in either of the series (5) or (6):—

$$m = \frac{-4i}{2i-1} = 0, -4, -\frac{8}{3}, -\frac{12}{5}, -\frac{16}{7}, -\frac{20}{9}, \ldots, -2; \quad (5)$$

$$m' = \frac{-4i}{2i+1} = 0, -\frac{4}{3}, -\frac{8}{5}, -\frac{12}{7}, -\frac{16}{9}, -\frac{20}{11}, \ldots, -2. \quad (6)$$

$$\frac{m}{2m+4} \text{ or } \frac{-m'}{2m'+4} = i = 0, 1, 2, 3, 4, 5, \ldots, \infty;$$

$$m+2 = n = \frac{-2}{2i-1}; \quad m'+2 = n' = \frac{2}{2i+1}.$$

Among the *singular features*, the negative sum equals the product, or $-(m+m') = mm' = \frac{16i^2}{4i^2-1}$; also $m' = \frac{-m}{1+m}$ and $m = \frac{-m'}{1+m'}$, for the same value of $i$. A change of the sign of $i$ likewise changes the value of $m$ from either one of the series (5), (6) to the other.

### RECIPROCAL TRANSFORMATION.

In equation (1) let us substitute $u = \frac{1}{u_1}$, and $x = x_1^{\frac{1}{m+1}}$; the result will still have Riccati's form:

$$\frac{du_1}{dx_1} + \frac{c}{m+1} u_1^2 = \frac{b}{m+1} x_1^{\frac{-m}{m+1}}. \quad (7)$$

By this operation, the given exponent $m$ is changed to $\frac{-m}{m+1}$, that is, from one of the series (5),(6) to the other, as from $-\frac{12}{7}$ to $-\frac{12}{5}$. If $u$ in (1) be large, its reciprocal in (7) will evidently be small, and conversely; since $uu_1 = 1$. To repeat the transformation (7) would merely return to the equation (1). It is remarkable that Riccati's form alike re-appears for the reciprocal of $u$ in (7), and for the reciprocal of $x$ in the following equation (8). For which, let us substitute in (1) $xz = 1$, or $x = \frac{1}{z}$, and $u = \frac{z}{b} + z^2 y$. The general result so found for the reciprocal of $x$, is also integrable at once when $m = -4$.

$$\frac{dy}{dz} - by^2 = -cz^{-m-4}. \quad (8)$$

DIRECT TRANSFORMATION.

By the usual substitution of $u = \frac{1}{bx} + \frac{1}{x^2 u_1}$, in equation (1), we find first

$$\frac{x^2 du_1}{dx} + cx^{m+4} u_1^2 = b.$$

This result is brought into Riccati's form by making $x = x_1^{\frac{1}{m+3}}$:

$$\frac{du_1}{dx_1} + b_1 u_1^2 = c_1 x_1^{m_1}. \qquad (9)$$

Here

$$b_1 = \frac{c}{m+3}, \quad c_1 = \frac{b}{m+3}, \quad m_1 = -\frac{m+4}{m+3},$$
$$b_2 = \frac{c_1}{m_1+3}, \quad c_2 = \frac{b_1}{m_1+3}, \quad m_2 = -\frac{m_1+4}{m_1+3}, \text{ etc.}$$

The process can thus be continued to any extent by alteration and substitution. When $m$ belongs in the series (5),

$$m = \frac{-4i}{2i-1}, \quad m_1 = -\frac{m+4}{m+3} = \frac{-4(i-1)}{2(i-1)-1}.$$

That is, the direct transformation from (1) to (9) has diminished $i$ by 1. Hence when $i$ is a positive integer, the repetition of the process $i$ times, would reduce the exponent to 0, which renders the equation integrable in the form of (2) or (3).

On the contrary when $m$ belongs in the series (6), we find

$$m' = \frac{-4i}{2i+1}, \quad m'_1 = -\frac{m'+4}{m'+3} = \frac{-4(i+1)}{2(i+1)+1}. \qquad (10)$$

Here each direct transformation increases $i$ by 1, or removes one step from left to right, as from $-\frac{8}{5}$ to $-\frac{12}{7}$. Hence if the given exponent $m$ occurs in (6), the reciprocal transformation will first change it to (5) where direct transformations will reduce the exponent to 0. By these or similar operations, the integrable cases were first resolved by continued fractions, which have opened the way to the complete solution.

INTEGRATION.

Comparing equation (1) with the linear equation $\frac{du}{dx} + Pu = Q$, let us equate the second terms, making $bu^2 = Pu$, or $bu = P$. The well known formula for

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solving the linear equation contains an exponential, here denoted by $w$, such that $\int Pdx = \log w$, or $Pdx = \frac{dw}{w}$. Therefore equating the two expressions for $P$, and substituting the result in equation (1) we find, by making $m + 2 = n$,

$$u = \frac{dw}{bwdx}, \quad x^2 \frac{d^2w}{dx^2} = bcx^n w. \tag{11}$$

To discover the form of the integral, we adopt the usual method of first placing $\frac{d^2w_0}{dx^2} = 0$, whence by integration $w_0 = Px + Q$.

To satisfy equation (11), these constants $P$, $Q$ may next be regarded as variables or variable series, assumed as follows: If $w_1 = Px$, and $w_2 = Q$,

$$w_1 = C_1 x [1 + A\ (bcx^n) + B\ (bcx^n)^2 + \ \ldots\ ],$$
$$w_2 = C_2\ [1 + A'(bcx^n) + B'(bcx^n)^2 + \ \ldots\ ].$$

The two series represent two "particular solutions," as the theory requires. Each series, as $w_1$, is to be substituted separately in equation (11) to determine $A, B, \ldots$ by the common "method of indeterminate co-efficients."

$$w_1 = C_1 x \left\{ 1 + \frac{bcx^n}{n(n+1)} + \frac{(bcx^n)^2}{n(n+1)\,2n(2n+1)} + \frac{(bcx^n)^3}{n(n+1)\,2n(2n+1)\,3n(3n+1)} + \ \ldots \right\},$$

$$w_2 = C_2\ \left\{ 1 + \frac{bcx^n}{n(n-1)} + \frac{(bcx^n)^2}{n(n-1)\,2n(2n-1)} + \frac{(bcx^n)^3}{n(n-1)\,2n(2n-1)\,3n(3n-1)} + \ \ldots \right\},$$

$$w = w_1 + w_2, \quad u = \frac{\frac{dw_1}{dx} + \frac{dw_2}{dx}}{b(w_1 + w_2)}. \tag{12}$$

This integral in series is entirely regular and convergent, but it has its failing cases. For when $n - 1 = 0$, or $2n - 1 = 0$, etc., the corresponding terms become infinite, and the arbitrary constant divides out from $u$.

ANOTHER FORM OF INTEGRAL.

Let the independent variable $x$ be changed to $z$ in equation (11) by making

$$x = z^{\frac{2}{n}},$$
$$\frac{dw}{dx} = \frac{n}{2} z^{\frac{m}{n}} \frac{dw}{dz},$$

$$\frac{x^2d^2w}{dx^2} = \frac{n^2z^2}{4}\frac{d^2w}{dz^2} + \frac{mnz}{4}\frac{dw}{dz},$$

$$\frac{d^2w}{dz^2} + \frac{2i}{z}\frac{dw}{dz} + q^2w = 0. \qquad (13)$$

Here $\frac{m}{n} = 2i$, where $i$ may be positive or negative, and $-\frac{4bc}{n^2} = q^2$. Let us first suppose $b,c$ in equation (1) to have unlike signs, so that $q^2$ is essentially positive. As equations (3) and (12) suggest, let us assume

$$w = v\cos(qz + C) + v'\sin(qz + C),$$

$$\frac{dw}{dz} = \left(\frac{dv}{dz} + v'q\right)\cos(qz + C) + \left(\frac{dv'}{dz} - vq\right)\sin(qz + C).$$

Substituting for $w$ in equation (13), the total co-efficient of $\cos(qz + C)$ becomes zero, as authorized by the two unknown quantities $v$ and $v'$; and this reduces the co-efficient of $\sin(qz + C)$ to zero. Thus we find the two simultaneous equations

$$\frac{d^2v'}{dz^2} - 2q\frac{dv}{dz} + \frac{2i}{z}\left(\frac{dv'}{dz} - qv\right) = 0, \quad \frac{d^2v}{dz^2} + 2q\frac{dv'}{dz} + \frac{2i}{z}\left(\frac{dv}{dz} + qv'\right) = 0.$$

Let $v'' = v' + kv$. Multiplying the latter of the two equations by $k$ and adding to the former, we find when $k = \sqrt{-1}$,

$$\frac{d^2v''}{dz^2} + \left(\frac{2i}{z} + 2qk\right)\frac{dv''}{dz} + \frac{2iqk}{z}v'' = 0. \qquad (14)$$

The expression for $w$ above contains one arbitrary constant $C$, so that only one particular solution of (14) will be required to give the other $C'$ by "the method of indeterminate co-efficients," assuming

$$v'' = v' + kv = C'z^{-i}\left\{1 + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3} + \ldots\right\}.$$

Substituting in 14,

$$B = \frac{-i(i-1)}{1\,.\,2qk}, \quad C = \frac{-B(i+1)(i-2)}{2\,.\,2qk}, \quad D = \frac{-C(i+2)(i-3)}{3\,.\,2qk}, \text{ etc.}$$

The real terms of the series evidently give the value of $v'$, and the imaginary terms divided by $k$ give the value of $v$. And thus,

$$w = C'z^{-i}\left\{\frac{i(i-1)}{1\,.\,2qz} - \frac{(i+2)\ldots(i-3)}{1\,.\,2\,.\,3\,(2qz)^3} + \ldots\right\}\cos(qz + C) \qquad (15)$$

$$+ C'z^{-i}\left\{1 - \frac{(i+1)\ldots(i-2)}{1\,.\,2\,(2qz)^2} + \frac{(i+3)\ldots(i-4)}{1\,.\,2\,.\,3\,.\,4\,(2qz)^4} - \ldots\right\}\sin(qz + C).$$

Referring now to the first of equations (11) and to subsequent derivatives of $x$ and $w$ with respect to $z$, we next find equation (16); and the quotient of (16) divided by (15) determines $u$, thus: —

$$\frac{dw}{bdx} = C' \frac{qnz^i}{2b} \times \tag{16}$$

$$\left\{ \begin{aligned} &\left\{ 1 - \frac{(i+2)\ldots(i+1)}{1\,.\,2\,(2qz)^2} + \frac{(i+4)\ldots(i-3)}{1\,.\,2\,.\,3\,.\,4\,(2qz)^4} - \ldots \right\} \cos(qz + C) \\ &- \left\{ \frac{i(i-1)}{1\,.\,2qz} - \frac{(i+3)\ldots(i-2)}{1\,.\,2\,.\,3\,(2qz)^3} + \ldots \right\} \sin(qz + C) \end{aligned} \right\};$$

$$\frac{du}{dx} + bu^2 = cx^m; \tag{1}$$

$$m + 2 = n, \quad \frac{m}{n} = 2i, \quad u = \frac{dw}{bdx} \div w. \tag{17}$$

Also $z = x^{\frac{n}{2}}$, and $q^2 = \frac{-4bc}{2n}$. And the signs of $b, c$ in equations (1), (15), and (16) are supposed to be *unlike* in the integral (17).

In case $b, c$ in (1) have *like signs* the requisite changes in (15) and (16) are easily made; let $q = q'\sqrt{-1}$, and let $C'\sqrt{-1}$ be written in place of the arc $C$; also make this $(q'z + C')\sqrt{-1} = t\sqrt{-1}$. By changing the former sine and cosine to their exponential values, we have

$$\sin(qx + C) = \frac{\varepsilon^{-t} - \varepsilon^{+t}}{2\sqrt{-1}}, \quad \cos(qx + C) = \frac{\varepsilon^{-t} + \varepsilon^{+t}}{2}.$$

Substituting the new equivalents, and then dividing (16) by (15), we find that the factor $\sqrt{-1}$ first renders all the terms of $2q'z$ positive, and then disappears, leaving, when $b$ and $c$ have like signs, and $q'z + C' = t$,

$$\begin{aligned} N = &\left\{ 1 + \frac{(i+2)\ldots(i-1)}{1\,.\,2\,(2q'z)^2} + \ldots \right\} (\varepsilon^{-t} + \varepsilon^{+t}), \\ &- \left\{ \frac{i(i+1)}{2q'z} - \ldots \right\} (\varepsilon^{-t} - \varepsilon^{+t}), \\ J = &\left\{ 1 + \frac{(i+1)\ldots(i-2)}{1\,.\,2\,(2q'z)^2} + \ldots \right\} (\varepsilon^{-t} + \varepsilon^{+t}), \\ &+ \left\{ \frac{i(i-1)}{2q'z} + \ldots \right\} (\varepsilon^{-t} - \varepsilon^{+t}), \\ u = &- \frac{nq'z^{2i}}{2b} \frac{N}{J}. \end{aligned} \tag{18}$$

Thus the integral of Riccati's Equation is one of comparative simplicity. For by adding 1 to each factor of $i$ in the denominator, or by simply changing the sign of $i$ in the factors, we have the series of the numerator. The series are

evidently finite when $i$ is an integer; but for all other values of $i$, they are infinite, and ultimately divergent, like well known series for the Gamma function. As the convergent parts of the latter function have long been used continuously, so may a limited number of terms of the Riccati series be correctly used for computing the true value of $u$, when $i$ falls between the integer values.

[NOTE.—This important inference is further confirmed by the fact that an interpolation formula could certainly be so employed, and that the finite part of the series for $w$ in equation (15) has precisely the form of an interpolation formula, deduced from actual values of $u$. For example, when $i - 4 = 0$, the series for $w$ in equation (15) reduces to four terms, and the series (16) to five terms, which will determine the value of $u$, not only when $i = 4$, but also when $i = 3.2$, $3.3$, $3.7$, $3.8$, or any value intermediate between 3 and 4. By first augmenting $i$, as shown by equation (10), the number of terms may be augmented, and $u$ computed to any degree of accuracy.]

## A DISCUSSION OF THE EQUATION OF THE SECOND DEGREE IN TWO VARIABLES.

BY PROF. O. H. MITCHELL, Marietta, Ohio.

[CONCLUDED FROM PAGE 83.]

### ELLIPSES CLASSIFIED.

($\alpha$). $R's\Delta < 0$. Since $\Delta$ is supposed negative, $s$ is here positive. Distinguishing by the subscripts 1 and 2 the transverse and conjugate values, we have

$$e_1^2 = \frac{2R'}{R' + s}, \quad e_2^2 = \frac{2R'}{R' - s},$$

$$p_1'' = \sqrt{\frac{-4\Delta}{R'(s - R')^2}}, \quad p_2'' = \sqrt{\frac{+4\Delta}{R'(s + R')^2}}.$$

Whence it is seen that $e_1$ and $p_1''$ are real, while $e_2$ and $p_2''$ are imaginary. Hence, since to the real directrices and foci there corresponds a real value of $e$, the curve is real. $A_1$ and $A_2$ are both real and finite, and (15) shows that the curve has no point at a distance from the centre greater than $A_1$ or less than $A_2$. The two values of the semi-parameter are both real. This variety of the ellipse may therefore be described as having *real, finite, unequal axes.*

($\beta$). $R's\Delta > 0$. This implies that $s$ is negative. Hence $e_1$ and $p_2''$ are imaginary while $e_2$ and $p_1''$ are real. Hence the real value of the ratio $e$ corresponds to the imaginary directrices while the imaginary value of $e$ corresponds to

the real directrices. Thus the curve can have no real point, and is called an *imaginary ellipse*. $A_1$ and $A_2$, $P_1$ and $P_2$, are all imaginary.

($\gamma$). $R'sJ = 0$. Since $h^2 - ab < 0$, $s$ can not vanish. We have then three cases.

($a$). $R' = 0, J < 0$. This implies $h = 0$ and $a = b$. Hence $e_1^2 = e_2^2 = 0$, and $p_1'' = p_2'' = \infty$. Thus the four directrices lie at an infinite distance from the centre. From (14) it is seen that the four foci coincide with the centre. $A_1 = A_2 = \sqrt{\frac{-2sJ}{s^2a^2}}$. The semi-axes will therefore be real if $sJ < 0$, and imaginary if $sJ > 0$. Equation (15) becomes

$$x^2 + y^2 = A^2.$$

The curve is a *circle*, real if $A^2$ is positive ($sJ < 0$), imaginary if $A^2$ is negative ($sJ > 0$).

($b$). $R' > 0, J = 0$. The metrical results (14) all vanish in this case, or say become infinitesimal. Equation (16) becomes

$$(s - R')x^2 + (s + R')y^2 = 0.$$

The curve is, properly speaking, an *ellipse with unequal infinitesimal semi-axes*, since $A_1^2 : A_2^2 = (s - R') : (s + R')$.

($c$). $R' = 0, J = 0$. This case is the same as the last except that the semi-axes are equal, and $e^2 = 0$. The locus is a *circle with infinitesimal radius*.

HYPERBOLAS CLASSIFIED.

($\alpha$). $R'sJ < 0$. This implies that $s$ is positive. Hence $p_1''$ is real while $p_2''$ is imaginary, $1 < e_1^2 < 2$, $e_2^2 < 2$, and $A_1^2 > -A_2^2$, $A$ being real, while $A_2$ is imaginary. Equation (15) becomes

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1,$$

where $A_2^2$ is negative. So that whatever real value $y$ may have from $+\infty$ to $-\infty$, $x$ has a corresponding real value.

($\beta$). $R'sJ > 0$. Here $s$ is negative, $e_1^2 > 2$, $1 < e_2^2 < 2$, and $A_1^2 < -A_2^2$. Otherwise this curve is the same as the last.

($\gamma$). $R'sJ = 0$. Since $h^2 - ab > 0$, $R'$ can not vanish. We have three cases.

($a$). $s = 0, J < 0$. In this case the metrical results (14) are such that each transverse quantity is equal to the corresponding conjugate quantity multiplied by $-\sqrt{-1}$. We have $e_1^2 = e_2^2 = 2$. Since $A_1^2 = -A_2^2$, equation (15) becomes

$$x^2 - y^2 = A^2.$$

This curve is called an *equilateral hyperbola*. The semi-parameters are respectively equal to the semi-axes in analogy with the circle.

(*b*). $s \neq 0$, $\Delta = 0$. Here the metrical results (14) all vanish or, say, become infinitesimal. The foci are at the centre, the directrices pass through the centre (and are all real), and the semi-axes and semi-parameters are infinitesimal. The locus may be considered as an hyperbola with *unequal infinitesimal axes*, for $A_1^2 : A_2^2 = (s - R') : (s + R')$. Strictly speaking neither axis can be distinguished as transverse. Equation (16) becomes

$$(s - R')\, x^2 + (s + R')\, y^2 = 0,$$

which, since $R' > s$, is factorable into two equations of the first degree.

(*c*). $s = 0$, $\Delta = 0$. This case is the same as the last except $A_1^2 = -A_2^2$ and $e_1^2 = e_2^2 = 2$. The locus is an *equilateral hyperbola with infinitesimal axes* whose equation is

$$x^2 - y^2 = 0.$$

PARABOLAS CLASSIFIED.

Since $h^2 = ab$, $a$ and $b$ are of the same sign and $\Delta = -(af^2 - 2fgh + bg^2)$ Hence when $\Delta$ is made negative $s$ is positive. We have therefore

$$e_1^2 = 1, \quad e_2^2 = \infty,$$

for every case of the parabola.

(*a*). $R's\Delta < 0$. From (14) we get

$$p_1'' = \infty,$$
$$p_2'' = \sqrt{\frac{\Delta}{s^3}},$$
$$A_1 = \infty,$$
$$A_2 = \infty,$$

Foci to centre $= \infty$,

Focus to transverse directrix $= \dfrac{af - gh}{a^{\frac{1}{2}} s^{\frac{3}{2}}}$,

Focus to conjugate directrix $= \infty$,

Transverse semi-parameter $= \dfrac{af - gh}{a^{\frac{1}{2}} s^{\frac{3}{2}}}$,

Conjugate semi-parameter $= \infty$.

Although $A_1$ and $A_2$ are infinite, their ratio $\sqrt{s - R'} : \sqrt{s + R'}$ is infinitesimal. This curve is named a *true parabola*.

The co-ordinates of the centre (12) and (13) are infinite. From (7) we get

$$\theta' \sqrt{b}\, x + \sqrt{a}\, y = \frac{f^2 + g^2 - cs}{\sqrt{af} + \theta' \sqrt{bg} \pm (\sqrt{af} + \theta' \sqrt{bg})},$$

as the equations of the two real directrices of the parabola, of which one is seen

to be within a finite distance from the origin, the other infinitely distant. We also get

$$-\theta'\sqrt{a}\,x + \sqrt{b}\,y = \frac{f^2 + g^2}{\theta'\sqrt{ag} - \sqrt{bf} \pm \sqrt{-(\sqrt{af} + \theta'\sqrt{bg})^2}}$$

as the equations of the two imaginary directrices, of which both are at a finite imaginary distance from the origin. These results are obtained from the first form of the value of $p$, the second being indeterminate.

The co-ordinates of the focus corresponding to that real directrix which is within a finite distance from the origin are

$$x' = \frac{b(f^2 - g^2 - cs) + 2fgh}{2s(bg - fh)},$$

$$y' = \frac{h(f^2 - g^2 + cs) - 2bfg}{2s(bg - fh)},$$

and the co-ordinates of the other three foci are infinite. Since both values of $p_2$ are imaginary, it follows that both values of $x_2'$, $y_2'$ are imaginary though infinite.

From (9) we get as the equations of the axes of the parabola

$$hx + by + \frac{bf + gh}{s} = 0,$$

$$hx - ay + \infty = 0,$$

the first being that of the transverse axis.

Equation (15) is unmanageable for the parabola, the co-efficients being infinite, but (16) does not present this difficulty. It becomes

$$y^2 = \pm 2P_1 x.$$

The second member has an ambiguous sign for $\frac{s - R'}{s + R'} = 0$. This means that neither vertex can be considered as the "*left hand vertex*," for each is infinitely distant from the other either to the right or to the left of it. The equation becomes $y^2 = 2P_1x$ if the origin be taken at that vertex from which the curve proceeds to the right, and $y^2 = -2P_1x$ for the other vertex.

The form conjugate to (16) becomes, after multiplying through by $s - R'$,

$$(s + R')\,x^2 = \pm\sqrt{\frac{-8\Delta}{s - R'}}\,x,$$

an equation which breaks up into the two equations

$$x = 0 \text{ and } (s + R')\,x = \pm\sqrt{\frac{-8\Delta}{s - R'}}.$$

This means that from the point of view of a conjugate vertex a parabola appears as two parallel straight lines infinitely distant from each other.

($\beta$). $R's\Delta > 0$. This is impossible, since $s$ is positive if $\Delta$ is negative.

($\gamma$). $R's\Delta = 0$. We have to consider three cases.

($a$). $R's = 0$, $\Delta \neq 0$. These are impossible conditions, for the first implies $a = b = h = 0$; hence $\Delta$ can not be different from zero.

($b$). $R's = 0$, $\Delta = 0$. These conditions imply $a = b = h = 0$, and this case has already been noticed as that of a straight line.

($c$). $R's \neq 0$, $\Delta = 0$. The most elementary way of determining the locus in this case is to put ($B$) into the form

$$(ax + hy + g)^2 - [(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)] = 0,$$

by the process of completing the square. Then we see that $h^2 - ab = 0$ and $\Delta = 0$ reduce the equation to

$$(ax + hy + g)^2 - (g^2 - ac) = 0, \qquad (17)$$

which is seen to be the equation of two parallel lines at a distance from each other equal to

$$2\sqrt{\frac{g^2 - ac}{as}}.$$

These two lines will be real and different if $g^2 - ac > 0$,
" " coincident " " $= 0$,
imaginary " " $< 0$.

In the same way we obtain

$$(by + hx + f)^2 - (f^2 - bc) = 0$$

as the equation of the lines, and

$$2\sqrt{\frac{f^2 - bc}{bs}}$$

as the distance between them. Whence $f^2 - bc$ might replace $g^2 - ac$ in the conditions given above. Hence the conditions in symmetric form are

$$f^2 + g^2 - cs > 0,$$
$$f^2 + g^2 - cs = 0,$$
$$f^2 + g^2 - cs < 0,$$

according as the lines are real and different, coincident, or imaginary.

Applying (9) to find the equations of the axes of this locus we get

$$hx + by + f = 0,$$
$$-hx + ay + \frac{0}{0} = 0;$$

the first being the midway parallel between the lines, the second at right angles to this at an indeterminate distance from the origin. Hence the centre of the locus is found to be indeterminate as also appears from the co-ordinates of the centre.

Since $J = 0$ we get from (7)

$$p_1 = \frac{f^2 + g^2 - cs}{0 \pm 0}$$

$$p_2 = -\frac{f}{bs \pm 0} = -\frac{g}{as \pm 0}.$$

Hence two directrices are coincident at an infinite distance from the origin, the other two coincident at a finite distance. Strictly speaking neither axes can be said to be transverse. But, assuming that $s$ has been made positive, we shall call those results transverse which come from the substitution of the positive value of $R$. Hence the midway parallel between the lines is the transverse axis and also the conjugate directrix. Hence

$$A_1 = \infty, \quad A_2 = \sqrt{\frac{g_2 - ac}{as}} = \sqrt{\frac{f - bc}{bs}};$$

$$e_1 = 1, \quad e_2 = \infty.$$

An examination of the co-ordinates of the foci shows that they all lie at infinity.

Returning now to the original equations (1) to (6) from which we obtained the values of $e, \lambda, \mu, p, x', y'$, let us inquire how many solutions there ought to be. Dividing each of the first five equations of the set by equation (6), we get rid of the multiplier $\rho$. We then have, putting $k = x'^2 + y'^2 - e^2p^2$,

$$\begin{aligned}
\frac{1 - e^2\lambda^2}{k} &= \frac{a}{c}, \\
\frac{-e^2\lambda\mu}{k} &= \frac{h}{c}, \\
\frac{1 - e^2\mu^2}{k} &= \frac{b}{c}, \\
\frac{e^2p\lambda - x'}{k} &= \frac{g}{c}, \\
\frac{e^2p\mu - y'}{k} &= \frac{f}{c};
\end{aligned} \tag{18}$$

a system of five equations, each of the fourth degree in the quantities $e, \lambda, \mu, p, x', y'$. Joining to these the sixth equation $\lambda^2 + \mu^2 = 1$, the system of six equations have $2 \cdot 4^5$ or 2048 solutions. That is, a conic has 2048 each of directrices foci, and eccentricities, when considered from the algebraic point of view.

But our process of elimination yielded only sixteen solutions. We found

two values of $p, x', y'$, for each of the two values of $e^2$, and two values of $\lambda$ and $\mu$ for each pair of values of $p$. We rejected the negative value of $\mu$ as leading to the same geometric results as its positive value. We thus had left eight solutions of the system of equations. The negative value of $e$ may also be rejected as not giving essentially different geometric results. There will be left four solutions. Since the process of elimination consisted only in the addition, subtraction, and multiplication of the equations (division of equations not being employed), it follows that no set of finite solutions could have escaped. The remaining 2032 sets of solutions must therefore be partly or wholly infinite whatever may be the values of $a, b, c, f, g, h$. That the set of equations (17) may have a set of values of $e$, $\lambda, \mu, p, x', y'$ partly infinite appears as follows. Suppose $a, b, c, f, g, h$ to have any finite values, then if $\lambda, \mu, p$ be supposed finite, and $e, x', y'$ be supposed infinities of the first order the equations may be satisfied. Furthermore, if $e, \lambda, \mu, p$ be supposed infinites of the first order, and $x', y'$ infinites of the second order, the equations may be satisfied.

## THE HOMOGENEOUS EQUATION OF FOUR TERMS TO EXPRESS THE HOMOGRAPHIC DIVISION OF TWO STRAIGHT LINES.

By Mr. R. D. Bohannan, University of Virginia.

Chasles in his *Traité de Géométrie Supérieure* Art. 137, (Ed. 1852),* says: If two straight lines are divided homographically, and we take upon the one line two fixed points $a$, $c$, and upon the other, two fixed points $b'$, $d'$, homographic division will be expressed by the homogeneous equation

$$\frac{am}{cm}\cdot\frac{b'm'}{d'm'} + \lambda\frac{am}{cm} + \mu\frac{b'm'}{d'm'} + \nu = 0.$$

His first demonstration (Art. 137) is indirect, and would not readily suggest itself, and the second demonstration (Art. 140) is cumbrous. The proposition may also be demonstrated in the following manner:—

By Art. 133, homographic division is expressed by the following equation:—

$$am\,.\,b'm' - am\,.\,b'J' - b'm'\,.\,aI + aI\,.\,b'a' = 0,$$

in which $I$ is the point on the first line corresponding to infinity on the second and $J'$ the point on the second corresponding to infinity on the first. Dividing

*Art. 137, Ed. 1852 is 144, Ed. 1880; 133 is 139; 140 has been omitted.

by $aI \,.\, b'J'$, we have:—

$$\frac{am}{aI} \cdot \frac{b'm'}{b'J'} - \frac{am}{aI} - \frac{b'm'}{b'J'} + \frac{b'a'}{b'J'} = 0.$$

Introducing the point at infinity on each line,

$$\left(\frac{am}{aI} \div \frac{\infty\, m}{\infty\, I}\right)\left(\frac{b'm'}{b'J'} \div \frac{\infty\, m'}{\infty\, J'}\right) - \frac{am}{aI} \div \frac{\infty\, m}{\infty\, I} - \frac{b'm'}{b'J'} \div \frac{\infty\, m'}{\infty\, J'} + \frac{b'a'}{b'J'} \div \frac{\infty\, a'}{\infty\, J'} = 0.$$

This is an equation connecting $a, m, I, \infty$ of the first line and $b', m', \infty, J'$ of the second. While the three points $m, I, \infty$ correspond to the three, $m', \infty, J'$, the point $a$ does not correspond to the point $b'$. We may, therefore, fix $a$ and $b'$ arbitrarily. The equation involves only anharmonic ratios; thus (Art. 20) it remains true when for the corresponding points, $m, I, \infty$; $m', \infty, J'$, we substitute the corresponding points, $m, d, c$; $m', d', c'$. It is evident that we may fix $c, c'$ and $d, d'$ arbitrarily as corresponding points. Hence the preceeding equation becomes

$$\left(\frac{am}{ad} \div \frac{cm}{cd}\right)\left(\frac{b'm'}{b'c'} \div \frac{d'm'}{d'c'}\right) - \frac{am}{ad} \div \frac{cm}{cd} - \frac{b'm'}{b'c'} \div \frac{d'm'}{d'c'} + \frac{b'a'}{b'c'} \div \frac{d'a'}{d'c'} = 0,$$

in which $a, c$ are any two fixed points on the one line and $b', d'$ any two fixed points on the other. We may write the equation thus:—

$$\frac{am}{cm} \cdot \frac{b'm'}{d'm'} \cdot \frac{cd}{ad} \cdot \frac{d'c'}{b'c'} - \frac{am}{cm} \cdot \frac{cd}{ad} - \frac{b'm'}{d'm'} \cdot \frac{d'c'}{b'c'} - \frac{b'a'}{d'a'} \cdot \frac{d'c'}{b'c'} = 0.$$

Dividing this equation by $\frac{cd}{ad} \cdot \frac{d'c'}{b'c'}$, we have:—

$$\frac{am}{cm} \cdot \frac{b'm'}{d'm'} - \frac{am}{cm} \cdot \frac{b'c'}{d'c'} - \frac{b'm'}{d'm'} \cdot \frac{ad}{cd} + \frac{b'a'}{d'a'} \cdot \frac{ad}{cd} = 0$$

or

$$\frac{am}{cm} \cdot \frac{b'm'}{d'm'} + \lambda \frac{am}{cm} + \mu \frac{b'm'}{d'm'} + \nu = 0,$$

the required equation.

Comparing the last two equations, we have the values of $\lambda, \mu, \nu$ deduced in Art. 138.

## NOTE ON EQUATION VII, SECTION 8 OF GAUSS'S THEORIA MOTUS.

By Prof. Ormond Stone, University of Virginia.

Put $$\tan\frac{1}{2}a\tan\frac{1}{2}b\tan\frac{1}{2}c=1. \qquad (1)$$

By trigonometry, $$\tan^2\frac{1}{2}a=\frac{1-\cos a}{1+\cos a}, \text{ etc.}$$

Squaring, substituting, and clearing of fractions, (1) becomes

$$(1-\cos a)(1-\cos b)(1-\cos c)=(1+\cos a)(1+\cos b)(1+\cos c);$$
$$\therefore \cos a+\cos b+\cos c+\cos a\cos b\cos c=0;$$

whence $$\cos a=\frac{-\cos b-\cos c}{1+\cos b\cos c}, \qquad (2)$$

and, since $\sin a=\sqrt{1-\cos^2 a}$,

$$\sin a=\frac{\sin b\sin c}{1+\cos b\cos c}. \qquad (3)$$

Put $a=v$, $b=90^\circ-\varphi$, $c=180^\circ-E$; and (1) becomes

$$\tan\frac{1}{2}v\tan(45^\circ-\frac{1}{2}\varphi)\cot\frac{1}{2}E=1,$$

which is equivalent to Gauss's equation VII. Also, (2) and (3) become

$$\cos v=\frac{\cos E-\sin\varphi}{1-\sin\varphi\cos E},$$
$$\sin v=\frac{\cos\varphi\sin E}{1-\sin\varphi\cos E};$$

and since, on account of the symmetry of (1), $a$, $b$, and $c$ may be interchanged, we have also

$$\sin\varphi=\frac{\cos E-\cos v}{1-\cos v\cos E},$$
$$\cos\varphi=\frac{\sin v\sin E}{1-\cos v\cos E};$$

and
$$\cos E=\frac{\cos v+\sin\varphi}{1+\sin\varphi\cos v},$$
$$\sin E=\frac{\cos\varphi\sin v}{1+\sin\varphi\cos v}.$$

## THE NINE-POINTS CIRCLE.

By Mr. R. D. Bohannan, University of Virginia.

$ABC$ is a triangle. $A_a$ is the foot of the altitude from $A$. $A_m$ is the middle point of the side opposite $A$. The altitudes intersect at $O$. $A_0$ is the middle point of $AO$.

From similar triangles $BCC_a$, $BAA_a$,

$$\begin{aligned} BA_a : BC_a &:: BA : BC \\ &:: BC_m : BA_m ; \\ \therefore BA_a . BA_m &= BC_a . BC_m , \end{aligned}$$

and therefore $A_a$, $A_m$, $C_a$, $C_m$, lie on a circle.

From similar triangles $COA_a$, $CBC_a$,

$$\begin{aligned} CO : CA_a &:: CB : CB_a ; \\ \therefore CC_0 : CA_a &:: CA_m : CC_a ; \\ \therefore CC_0 . CC_a &= CA_a . CA_m ; \\ \therefore C_0 &\text{ lies on the circle } A_a, A_m, C_a, \end{aligned}$$

that is, $A_a, A_m, C_a, C_m, C_0$ lie on a circle;

Similarly, $A_a, A_m, C_a, C_m, A_0$ " " " "

$\therefore A_a, A_m, C_a, C_m, A_0, C_0$ " " " "

Similarly, $A_a, A_m, B_a, B_m, A_0, B_0$ " " " "

These two circles have three points in common;

$$\therefore A_a, A_m, A_0, B_a, B_m, B_0, C_a, C_m, C_0$$

lie all on the same circle.

---

## SOLUTIONS OF EXERCISES.

### 12

The result

$$\frac{p^2q^2 + 4p^3r - 8q^3 + 2pqr + 9r^3}{(r - pq)^2}$$

is given as the equivalent of the function

$$\left(\frac{\beta - \gamma}{\beta + \gamma}\right)^2 + \left(\frac{\gamma - \alpha}{\gamma + \alpha}\right)^2 + \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2 ,$$

where $\alpha$, $\beta$, $\gamma$ are the roots of the cubic

$$x^3 + px^2 + qx + r = 0.$$

Is this result correct? [*A. Hall.*]

SOLUTION.

The result is not correct, for if so then it must be true when $\beta = 0$, in which case $r = 0$, and the two expressions reduce to

$$\frac{8q}{p^2} - 1, \text{ and } \left(\frac{\gamma - \alpha}{\gamma + \alpha}\right)^2 + 2.$$

Now if these are identical, since $p^2 = (\gamma + \alpha)^2$ and $p^2 - 4q = (\gamma - \alpha)^2$, we have

$$\frac{8q}{p^2} - 1 = \frac{p^2 - 4q}{p^2} + 2;$$

or

$$3q = p^2,$$

which is not generally true. [*J. W. Nicholson.*]

## 14

A THIN inextensible cord in which the density of the material increases in geometric progression, as the distance from one end increases in arithmetic progression is laid directly across a rough horizontal cylinder the circumference of a normal section of which is equal to twice the length of the cord; to determine the co-efficient of friction, supposing the cord to be only just supported when its extremities are in the horizontal plane containing the axes of the cylinder. [*William Hoover.*]

SOLUTION.

Let $O$ be the centre of a right section, $APB$, of the cylinder. The string lies along $APB$, having its ends at $A$ and $B$, the smaller end being at $A$. Let $a$ be the radius of the cylinder. Call the density at $A$, unity. By the conditions of the problem, the density at $P$ is $e^{a\theta}$, $\theta$ being the angle $AOP$. Consider the equilibrium of an element, $PQ$, of the string. The string is about to move in the direction of $PQB$. Let $T$ be the tension at $P$, $T + dT$ that at $Q$, $R$ the normal reaction at $P$, and $\mu$ the co-efficient of friction. Resolving the forces acting on the element, $PQ$, along the tangent and normal at $P$, we have

$$dT - \mu R a\, d\theta - a g e^{a\theta} \cos\theta\, d\theta = 0,$$
$$T - R a\, d\theta - a g e^{a\theta} \sin\theta\, d\theta = 0.$$

Eliminating $R$, we have

$$\frac{dT}{d\theta} - \mu T = ag\epsilon^{a\theta}(\mu \sin\theta + \cos\theta),$$

a linear differential equation of the first order and degree. Multiplying by $\epsilon^{-\mu\theta}$

$$\frac{d}{d\theta}(T\epsilon^{-\mu\theta}) = ag\epsilon^{(a-\mu)\theta}(\mu \sin\theta + \cos\theta);$$

$$\therefore T = ag\epsilon^{\mu\theta}\int \epsilon^{(a-\mu)\theta}(\mu \sin\theta + \cos\theta)\, d\theta$$

$$= ag\epsilon^{\mu\theta}\left\{ C + \epsilon^{(a-\mu)\theta}\frac{(\mu(a-\mu)+1)\sin\theta + (a-2\mu)\cos\theta}{1+(a-\mu)^2}\right\}.$$

$T = 0$, when $\theta = 0$; $\quad\therefore C = \dfrac{2\mu - a}{1+(a-\mu)^2}$.

And $T = 0$, when $\theta = \pi$;

$$\therefore (2\mu - a) + \epsilon^{(a-\mu)\pi}(2\mu - a) = 0;$$

$$\therefore (2\mu - a) = 0, \text{ or } 1 + \epsilon^{(a-\mu)\pi} = 0.$$

The second equation is not satisfied by any real value of $\mu$. The first equation gives $\mu = \dfrac{a}{2}$. [*R. D. Bohannan.*]

## 14.

A FAISCEAU of parabolas can be drawn having the pole of a cardioid as the common focus, all passing through one point and all cutting the cardioid at right angles. [*H. A. Newton.*]

SOLUTION.

Take a faisceau of straight lines, all passing through the point, $(ai)$, in the plane of the complex variable $Z = x + yi$; also a faisceau of circles having the point, $(ai)$, as common centre.

The equation of any one of the straight lines is

$$y = mx + a. \qquad (1)$$

Being transformed by the function of the second degree,

$$w = Z^2, \text{ or } w + vi = (x + yi)^2 = x^2 - y^2 + 2xyi,$$

whence

$$w = x^2 - y^2,$$
$$v = 2xy,$$

$$x^2 = \frac{\pm\sqrt{w^2 + v^2} + w}{2},$$

$$y^2 = \frac{\pm\sqrt{w^2 + v^2} - w}{2};$$

equation (1) becomes

$$4m^2w^2 + 4(m^3 - m)wv + (m^4 - 2m^2 + 1)v^2 + 4a^2(m^2 - 1)w - 8a^2mv - 4a^4 = 0. \quad (2)$$

As $m$ varies from $\tan 0^\circ$ to $\tan \pi$, this equation represents a system of parabolas, all having the origin as focus and all passing through the point, $(w = -a^2)$.

The equation of any one of the circles in the variable plane is

$$x^2 + (y - a)^2 = r^2.$$

Being transformed by the same function, $w = Z^2$, this becomes

$$\sqrt{w^2 + v^2} = a\sqrt{2(\sqrt{w^2 + v^2} - w) + r^2} - a^2,$$

which reduces to

$$[w^2 + 2a^2w + v^2 + (a^2 - r^2)^2]^2 = 4r^4(w^2 + v^2). \quad (3)$$

Since the circles and straight lines with which we set out intersect at right angles, and since in the transformation employed similarity in the smallest parts is maintained; therefore, as $r$ varies, equation (3) represents a faisceau of curves, all of which cut the parabolas defined by (2) at right angles. Thus by this transformation a faisceau of parabolas is obtained having a common focus, all passing through one point and all cutting at right angles the curves defined by equation (3). All of these latter curves lie symmetrically about the axis of $w$.

When $r > a$, the curve is of the fourth order and forms a loop around the origin.

When $r < a$, it is still of fourth order, but does not form a loop.

When $r = \frac{a}{\sqrt{2}}$, the curve touches the imaginary axis at two points and lies wholly on the negative side of that axis.

When $r = a$, equation (3) becomes

$$(w^2 + 2a^2w + v^2)^2 = 4a^4(w^2 + v^2),$$

which is that of a cardioid, having the origin or common focus of the parabolas for its pole. This is the special case required by the problem.

[*A. G. Reeves.*]

## 15

IT IS required to inscribe a rectangle of breadth $nD$ within another rectangle whose diagonal is $B$. Show that the excess $E$ of the declivity $M$ of the diagonal of the greater rectangle over that of the length of the less may be calculated when $n$ is small, from the formula

$$\sin E = n \cos 2M [1 + 2n \sin 2M + n^2(3 - \cos 2M) + \ . \ . \ . \ ].$$

[*William M. Thornton.*]

SOLUTION.

If $m$ denote the declivity of the length of the inscribed triangle, we have two right angles whose corresponding sides

$$D \cos M - nD \sin m, \text{ and } nD \cos m;$$

$$D \sin M - nD \cos m, \text{ and } nD \sin m;$$

are proportional. Therefore

$$\frac{\cos M - n \sin m}{\sin M - n \cos m} = \frac{\cos m}{\sin m},$$

whence $$\sin (M - m) = n \cos 2m,$$

or $$\sin E = n \cos 2 (M - E),$$

or $$\sin E = n \cos 2M (1 + 2 \tan 2M \sin E - 2 \sin^2 E - 4 \tan 2M \sin E \sin^2 \tfrac{1}{2} E).$$

Accordingly we have to terms of the first order in $n$

$$\sin E_1 = n \cos 2M;$$

to terms of the second order

$$\sin E_2 = n \cos 2M (1 + 2n \sin 2M);$$

to terms of the third order

$$\sin E_3 = n \cos 2M [1 + 2n \sin 2M + n^2 (1 - 3 \cos 4M)],$$

and so on.

This is the well known problem of the design of the angle blocks for the Howe bridge-truss. [*W. M. Thornton.*]

## 20

A TUNNEL section, consisting of a rectangle surmounted by a semicircle, is required to accommodate a rectangle of breadth $B$ and height $H$. Show how to determine its proportion so that its (1) area, (2) perimeter shall be least.

[*William M. Thornton.*]

SOLUTION.

If $x$ denote the radius of the circular top, we shall find

1. For the perimeter

$$P = 2H + (\pi + 2)\,x - \sqrt{4x^2 - B^2};$$
$$\therefore \frac{dP}{dx} = \pi + 2 - \frac{4x}{\sqrt{4x^2 - B^2}};$$

whence for a minimum value of $P$

$$x = \frac{B}{2} \cdot \frac{\pi + 2}{\sqrt{\pi(\pi + 4)}}.$$

2. For the area

$$A = 2Hx + \tfrac{1}{2}\pi x^2 - x\sqrt{4x^2 - B^2};$$
$$\therefore \frac{dA}{dx} = 2H + \pi x - \frac{8x^2 - B^2}{\sqrt{4x^2 - B^2}} = 0$$

for a minimum value of $A$. This gives rise to a quartic which is solved most easily by trial.

Railway rolling stock of ordinary dimensions will require on a double-tracked road, with tracks $13'$ between centres, a rectangle $22'$ broad and $14'$ high above the bottom of the rails. Adding $1'$ for clearance we have

$$B = 24, \qquad H = 15;$$
$$\therefore x_1 = 13.326, \quad x_2 = 13.534;$$

corresponding to tunnel sections $26'.65 \times 22'.53$ and $27'.07 \times 22'.28$ respectively. To the second dimension the requisite addition for ballast and drain must, of course, be added. [*W. M. Thornton.*]

## 21

IF THE central force on a body moving in a parabola, latus rectum $4m$, were to cease acting at the vertex and continue interrupted till the body had described an angle of 60° about the focus; determine the orbit it would afterwards describe.

[*William Hoover.*]

SOLUTION.

We have for the point where the force again begins to act, the well known relations

$$\frac{V^2}{k^2} = \frac{p}{r^2 \sin^2 \phi} = \frac{2}{r} - \frac{1}{a} \qquad (a)$$
$$= \frac{2}{r_0} = \frac{2}{m}, \text{ by hypothesis,}$$

where $V$ is the velocity of the body in its orbit, $k$ the constant of the system, $\psi$ the angle between the tangent to the new orbit and the prolongation of the corresponding radius vector ($r$), $r_0$ the radius vector for the vertex of the old orbit, and $p$, $a$, and $e$ are the semi-parameter, semi-major axis, and eccentricity of the new orbit. For the given point we have, as may easily be seen,

$$r = 2m,$$
$$\psi = 30^\circ,$$
$$r^2 \sin^2 \psi = m^2;$$

whence, by substitution in ($a$), we see that the new orbit will be an hyperbola, in which $p = 2m$, $a = -m$, $e = 1.3$. [*Ormond Stone.*]

## EXERCISES.

### 26

DETERMINE the maximum right cone $B$ inscribed in a given right cone $A$, the vertex of $B$ being at the centre of the base of $A$. [*O. Root, Jr.*]

### 27

THE NUMBER of points common to three surfaces of the $m$th, $n$th, and $p$th degrees being in general $mnp$, find the co-ordinates of all the real and imaginary points of intersection of the three surfaces, $y^2 + x^2 + z^2 = 4x$, $x^2 = yz$, and $y^2 = x^4$. [*H. A. Newton.*]

### 28

GIVEN the perpendicular, median, and bisector issuing from one and the same vertex of a plane triangle and terminating in the opposite side, to construct the triangle and determine a formula for its area. [*Marcus Baker.*]

### 29

GIVEN a pair of points $A,B$, if $C,D$ are such that $OD \cdot OC = OA^2 = OB^2$ and $AOC = AOD$, prove that the pair $A,B$ bears similar relation to the pair $C,D$. Show, also, the existence of a pair $E,F$ which bears the same relation to each of the pairs $A,B$ and $C,D$. [*Wm. Woolsey Johnson.*]

## 30

INVESTIGATE formulæ for the logarithms of 13, 19, and 73 in terms of the logarithms of primes less than 13 and of $(N + 1/N)$ where $N = 132495$; 262143; 274625. [*F. H. Loud.*]

## 31

IN THE TRIANGLE $ABC$ draw $AD$ to the point $D$ in $BC$; then will

$$AB^2 \cdot DC + AC^2 \cdot BD = BC \cdot AD^2 + BD \cdot DC \cdot BC.$$

[*J. R. Spiegel.*]

## 32

A TRIANGLE $PQR$ is inscribed in triangle $ABC$. Determine the ratios in which $P, Q, R$ divide the sides $BC, CA, AB$ in order that $AQR, BRP, CPQ$ may be respectively $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, of $ABC$. [*W. M. Thornton.*]

SELECTED.—ALGEBRAIC EQUATIONS.

## 33

THE CUBIC $ax^3 + 3bx^2 + 3cx + d = 0$ has one or three real roots as its discriminant

$$D = (ad - bc)^2 - 4(b^2 - ac)(c^2 - bd)$$

is positive or negative. If $D$ is zero, two roots are in general equal each to

$$\sqrt{\frac{c^2 - bd}{b^2 - ac}}.$$

## 34

SOLVE by quadratics the equations below and interpret their roots geometrically: —

$$x^4 - x^3\sqrt{15} + 4x^2 - 1 = 0,$$
$$x^4 - x^3\sqrt{3} - 2x^2 + 2x\sqrt{3} - 1 = 0.$$

## 35

SOLVE by quadratics the quartic

$$x^4 + ax^3 + bx^2 + ad^{\frac{1}{2}}x + d = 0.$$

## 36

SOLVE the cubic

$$\sqrt[3]{x} + \sqrt[3]{x - 7} = \sqrt[3]{3x + 3},$$

by eliminating its commensurable root.

### 37

If $a, b, c, d, e$ be the roots of the quintic

$$x^5 + px^3 + qx^2 + rx + s = 0,$$

find the values of

$$a^2(de + eb + bc) + b^2(ea + ac + cd) + c^2(ab + bd + de) + d^2(bc + ce + ea) + e^2(cd + da + ab).$$

### 38

Show that among any three roots $a, b, c$ of a biquadratic, as $x^4 + px^2 + q = 0$, this relations holds:—

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a + b + c}.$$

CONICS.

### 39

On a chord $MN$ of an ellipse as diameter a circle is drawn which cuts the curve in $P$ and $Q$. Show that the ratio of the intercepts of $MN$ and $PQ$ on the axis major is $e^2 : 2 - e^2$.

### 40

The vertex of an hyperbola and one asymptote is fixed. Find the locus of the focus.

### 41

Find the equations to the conics which cut the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

orthogonally at all their common points.

### 42

From two points on $Ox$, equidistant from $O$, tangents are drawn to the conic

$$ax^2 + 2hxy + by^2 = 2x.$$

Find the locus of their points of concourse.

### 43

The polars of points on one of two equal circles relative to the other envelop a parabola.

### 44

Find the locus of the points from which tangents drawn to $Ax^2 + By^2 = 1$ are parallel to conjugates of $ax^2 + 2hxy + by^2 = 1$.

# CONTENTS.


| | PAGE. |
|---|---|
| Integration of Riccati's Equation. By LEVI W. MEECH, | 97 |
| A Discussion of the Equation of the Second Degree in Two Variables (*concluded*). By O. H. MITCHELL, | 103 |
| The Homogeneous Equation of Four Terms to express the Homographic Division of Two Straight Lines. By R. D. BOHANNAN, | 109 |
| Note on Equation VII, Section 8 of Gauss's Theoria Motus. By ORMOND STONE, | 111 |
| The Nine-Points Circle. By R. D. BOHANNAN, | 112 |
| Solutions of Exercises, | 112 |
| Exercises, | 118 |



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